

ON THE COUNTING OF HOLOMORPHIC DISCS IN TORIC FANO MANIFOLDS

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ABSTRACT. We first compute three-point open Gromov-Witten numbers of Lagrangian torus fibers in toric Fano manifolds, and show that they depend on the choice of three points, hence they are not invariants. We show that for a cyclic A-infinity algebras, such counting may be defined up to Hochschild or cyclic boundary elements. In particular we obtain a well-defined function on Hochschild or cyclic homology of a cyclic A-infinity algebra, which has invariance property under cyclic A-infinity homomorphism.

1. INTRODUCTION

Gromov-Witten invariants, which are obtained from the intersection theory of the moduli spaces of pseudo-holomorphic curves without boundary, have provided exciting links to various branches of mathematics. Open Gromov-Witten invariants from the intersection theory of pseudo-holomorphic curves with boundary, are still largely mysterious. Most of the works in this direction has been carried out in the case that Lagrangian submanifold is given as a fixed point set of anti-symplectic involution or in the Calabi-Yau case (See for example, Katz-Liu [15], Welschinger [20], Cho [5], Solomon [19], Pandharipande-Solomon-Walsher [18], Fukaya [11])

The fundamental difference between open and closed cases is that while the moduli spaces of closed pseudo-holomorphic curves carry fundamental cycles, but the moduli spaces of open pseudo-holomorphic curves do not carry fundamental cycles but only fundamental chains. As the chains do not have good intersection property, it is not clear in general how to define open Gromov-Witten invariants. For example, even in the case of anti-symplectic involution, it is not yet known how to define them as an invariant for dimension > 3 . In this paper, we analyze two examples, three-point open Gromov-Witten numbers of Lagrangian torus fibers in toric Fano manifolds and a simple case of an open-closed invariant.

On the other hand the moduli space of pseudo-holomorphic discs is the main ingredient of the construction of A_∞ -algebra of Lagrangian submanifolds by Fukaya, Oh, Ohta and Ono [12]. The coefficients of $m_k(x_1, \dots, x_k)$ operation are related to counts of pseudo-holomorphic discs which intersect x_1, \dots, x_k in a cyclic order at the boundary of a disc (without cyclic ordering, it corresponds to l_k operation of induced L_∞ -algebra). A priori, these m_k or l_k are by nature, chain level operations, and in general do not provide homological invariants.

In this paper, we give a sufficient condition on chains to be intersected at the boundary so that totality of the bubbling off a disc becomes zero. We show that for

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a cyclic A-infinity algebras, such counting may be defined up to Hochschild or cyclic boundary elements, which may provide an invariant counting in a generalized sense. In particular we obtain a well-defined function on Hochschild or cyclic homology of a cyclic A-infinity algebra, which has invariance property under cyclic A-infinity homomorphism. The simple idea is that the condition to cancel out all discs bubbled off from the intersection problem with a given cycle \mathfrak{x} , is related to Hochschild or cyclic boundary operations. This is a revised version of the unpublished manuscript arXiv:math/0604502 using the language of cyclic A_∞ -algebras.

2. THREE POINT INTERSECTION NUMBERS OF LAGRANGIAN TORUS FIBERS IN TORIC FANO MANIFOLDS

Let M be a toric Fano manifold. The Floer theory of Lagrangian torus fibers of M has been developed in [2],[8] and [13]. The main characteristic of the theory is that its homology and the product structure is determined by the holomorphic discs of Maslov index two. And the number of holomorphic discs intersecting generic three points has not been discussed there as it involves Maslov index 4 discs. In this section, we present elementary geometric argument to find such holomorphic discs. Namely, we show that the count of holomorphic discs passing through generic 3 points depends on the choice of 3 points, by an explicit computation.

We first consider the case of $(\mathbb{C}^n, (S^1)^n)$, and explain how to extend the result to the case of toric Fano manifolds with Lagrangian torus fibers later. For a Lagrangian submanifold $L \subset M$, we denote by $\mathcal{M}_3(\beta, J)^*$ the moduli space of simple J -holomorphic discs with three boundary marked points of homotopy class $\beta \in \pi_2(M, L)$. Recall that the moduli space $\mathcal{M}_3(\beta, J)^*$ has an expected dimension

$$\dim(L) + \mu(\beta) - \dim(PSL(2; \mathbb{C})) + 3 = n + \mu(\beta).$$

Here $\mu(\beta)$ is the Maslov index. We have an evaluation map $ev_3 : \mathcal{M}_3(\beta, J)^* \rightarrow L^3$ given by the evaluation at the marked points. We want to count the number of points in $ev_3^{-1}(p_0, p_1, p_2)$ where p_i 's are disjoint generic points in L . Hence, to have a zero dimensional preimage, we require that $\mu(\beta) = 2n$.

Consider \mathbb{C}^n and a torus L defined by

$$L := \{(z_1, \dots, z_n) \mid \forall i, |z_i| = 1\}.$$

We can also consider other tori by setting $|z_i|$ to be different positive real numbers than 1, and the following computation will remain true by modifying constants appropriately.

Note that $\pi_2(\mathbb{C}^n, L)$ is generated by n elements of whose Maslov indices are two, which we denote by β_1, \dots, β_n . They are homotopy classes of the maps $w_i : D^2 \rightarrow \mathbb{C}^n$ defined by $z \mapsto (1, \dots, 1, z, 1, \dots, 1)$ where z is located at the i -st entry.

Let us fix a homotopy class of holomorphic disc to be $\sum_{j=1}^n \beta_j$. It is easy to see that for other homotopy classes with $\mu = 2n$, there wouldn't be any intersection between holomorphic discs of such class and generic three points as one of the factor in \mathbb{C}^n above has to remain constant.

The holomorphic disc of class $\sum_{j=1}^n \beta_j$ may be written as

$$(e^{c'_1 i} \frac{z - \alpha'_1}{1 - \alpha'_1 z}, \dots, e^{c'_n i} \frac{z - \alpha'_n}{1 - \alpha'_n z}).$$

By using an automorphism of the domain D^2 and simplifying the first entry, the above map can be written as (for $z \in D^2$)

$$(z, e^{c_2 i} \frac{z - \alpha_2}{1 - \overline{\alpha_2} z}, \dots, e^{c_n i} \frac{z - \alpha_n}{1 - \overline{\alpha_n} z}) \in \mathbb{C}^n. \quad (2.1)$$

Now we consider three points on the torus L . By applying the torus action, we may assume that the first point is given by

$$p_0 := (1, \dots, 1).$$

Now, denote two generic points of L as

$$p_1 := (e^{\theta_1 i}, \dots, e^{\theta_n i}), \quad p_2 := (e^{\theta_{n+1} i}, \dots, e^{\theta_{2n} i}).$$

We may assume that $0 < \theta_j < 2\pi$ for all j and that θ_j 's are distinct by genericity. Suppose the holomorphic disc given by the equation (2.1) passes through p_0 , p_1 , and p_2 at the boundary. Then, the disc intersects p_0 when $z = 1$, intersects p_1 when $z = e^{\theta_1 i}$ and intersects p_2 at $z = e^{\theta_{n+1} i}$.

By plugging in $z = 1, e^{\theta_1 i}, e^{\theta_{n+1} i}$ to the equation (2.1), we obtain the following equations for each $j = 2, \dots, n$,

$$e^{c_j i} \frac{1 - \alpha_j}{1 - \overline{\alpha_j}} = 1. \quad (2.2)$$

$$e^{c_j i} \frac{e^{\theta_1 i} - \alpha_j}{1 - \overline{\alpha_j} e^{\theta_1 i}} = e^{\theta_j i}. \quad (2.3)$$

$$e^{c_j i} \frac{e^{\theta_{n+1} i} - \alpha_j}{1 - \overline{\alpha_j} e^{\theta_{n+1} i}} = e^{\theta_{n+j} i}. \quad (2.4)$$

Now, we fix j and solve the above equation (find α_j and c_j) to find the holomorphic disc (2.1) passing through p_0, p_1, p_2 . We first set

$$\frac{e^{\theta_1 i} - \alpha_j}{e^{-\theta_1 i} - \overline{\alpha_j}} = e^{2\delta_1 i}$$

since it has modulus one. Here δ_1 may be chosen as the complex argument (between 0 and 2π) of $e^{\theta_1 i} - \alpha_j$. Similarly, we set

$$\frac{1 - \alpha_j}{1 - \overline{\alpha_j}} = e^{2\delta_2 i}$$

where δ_2 is the complex argument of $1 - \alpha_j$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (as α_j is in the disc D^2).

The equation (2.2) implies that $2\delta_2 = -c_j$ modulo 2π . Now, the equation (2.3) can be rewritten as

$$e^{c_j i} \frac{e^{\theta_1 i} - \alpha_j}{e^{-\theta_1 i} - \overline{\alpha_j}} = e^{(\theta_1 + \theta_j) i}. \quad (2.5)$$

We rewrite this again using δ_1, δ_2 to obtain

$$e^{-2\delta_2 i} e^{2\delta_1 i} = e^{(\theta_1 + \theta_j) i}.$$

This implies that $2\delta_1 - 2\delta_2$ equals $(\theta_1 + \theta_j)$ modulo 2π .

We let A, B, C the points $1, e^{\theta_1 i}, \alpha_j$ respectively. Geometrically (usual counter clockwise) angle from a vector \overrightarrow{CA} to \overrightarrow{CB} is given by $\delta_1 - \delta_2$ from our choices of complex arguments.

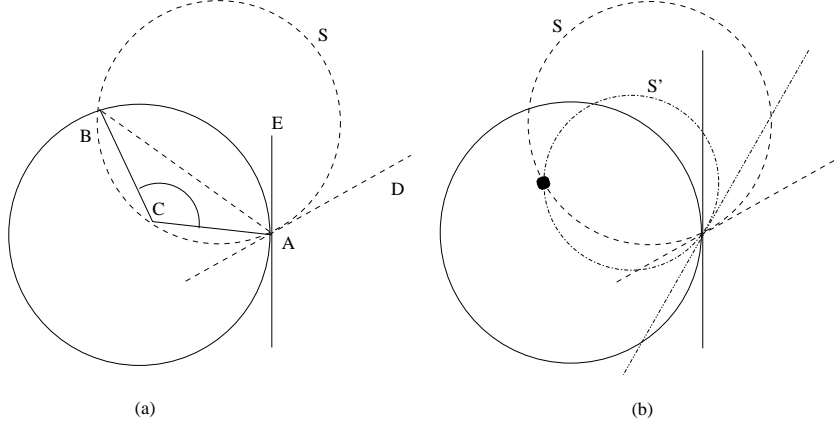


FIGURE 1. (a) Figure for the equation 2.3, (b) intersection of S and S'

Hence, we have $\delta_1 - \delta_2 = \frac{\theta_1 + \theta_j}{2}$ if $\theta_1 + \theta_j \leq 2\pi$ and otherwise $\delta_1 - \delta_2$ equals $\frac{\theta_1 + \theta_j}{2}$ or $\frac{\theta_1 + \theta_j}{2} - \pi$.

In any case, we observe from the elementary plane geometry that for any two points A, B on a plane, the set of all points C with a given fixed angle $\angle ACB$ forms a circle, say S , which passes through A and B (drawn as a dotted circle in the figure 1). Hence α_j lies on the circle S , or more precisely lies on the arc $S \cap D^2$.

Now consider the tangent line of the circle S at $A = 1$ and let D be a point on this line different from A . We also require that D and C to be located on the opposite side to each other with respect to the line \overline{AB} . Now we also consider the tangent line to ∂D^2 at 1, and denote by E the point $1 + i$ which lies on this line.

Lemma 2.1. *We have*

$$\angle BAE = \frac{\theta_1}{2}, \quad \angle DAE = \frac{\theta_j}{2}.$$

Proof. Note that the points $B = e^{\theta_1 i}$, $A = 1$, $E = 1 + i$ are already fixed. Hence, simple computation shows the first claim.

For the second claim, observe that the circle S is the excircle of the triangle ABC . Hence we have

$$\angle ACB = \angle DAB = \angle EAB + \angle DAE.$$

Actually, there can be two possibilities depending on whether C lies above the line AB or below the line AB . If C lies above the line AB we consider the $\angle ACB$ to be the one between π and 2π so that it equals the angle from \overrightarrow{CA} to \overrightarrow{CB} which is $\delta_1 - \delta_2$. This also holds if C lies below the line AB .

Hence we have

$$\angle DAE = \angle ACB - \angle EAB = \delta_1 - \delta_2 - \frac{\theta_1}{2}$$

which equals $\frac{\theta_j}{2}$ or $\frac{\theta_j}{2} - \pi$. But the latter is always negative, hence we obtain the second claim. \square

Hence the circle S on which α_j sits, passes through 1 and $e^{\theta_1 i}$ and intersects ∂D^2 at 1 with the angle $\theta_j/2$.

Now the third equation (2.4) can be solved in a similar way. The set of α_j which solves the equation (2.4) lies on a circle S' passing through 1 and $e^{\theta_{n+1}i}$ which intersect ∂D^2 at 1 with angle $\theta_{n+j}/2$.

To solve the (2.2) - (2.4) simultaneously for a fixed j is equivalent to find an intersection of two circles S and S' in the interior of D^2 , where the intersection point corresponds to α_j . Note that θ_i 's are given by p_0, p_1, p_2 and whether there exist an intersection of two circles S and S' in D^2 can be determined by the intersection angles θ_j, θ_{n+j} of the two circles with ∂D^2 .

More precisely, one can check as in Figure 1 (b) that if $0 < \theta_n < \theta_1$, and $0 < \theta_{n+j} < \theta_j$, then the circles intersect at one interior point, or if $0 < \theta_1 < \theta_n$, and $0 < \theta_j < \theta_{n+j}$, then the circles intersect at one interior point. In other cases, the circles do not intersect, which implies that there is no solution for the system of equations (2.2 - 2.4).

This can be summerized as the following proposition.

Proposition 2.2. *Let*

$$p_0 = (e^{\theta_{01}i}, \dots, e^{\theta_{0n}i}), p_1 = (e^{\theta_{11}i}, \dots, e^{\theta_{1n}i}), p_2 = (e^{\theta_{21}i}, \dots, e^{\theta_{2n}i})$$

be generic three points on the torus $L \subset \mathbb{C}^n$. Then if the cyclic ordering of three complex numbers $(e^{\theta_{10}i}, e^{\theta_{20}i}, e^{\theta_{30}i})$ on the unit circle $S^1 \subset \mathbb{C}$ agrees with the cyclic ordering of $(e^{\theta_{1j}i}, e^{\theta_{2j}i}, e^{\theta_{3j}i})$ for all $j = 1, \dots, n$, then there is a holomorphic disc $(D^2, \partial D^2) \rightarrow (\mathbb{C}^n, L)$ of the homotopy class $\sum_{j=1}^n \beta_j$, which passes through p_0, p_1 and p_2 . Otherwise, there does not exist a holomorphic disc of the given homotopy class which passes through these three points.

In the case of toric Fano manifolds, we proceed in the following way. Let Σ be a complete n -dimensional fan of regular cones defining a toric manifold M . Let $\{v_1, \dots, v_n\}$ be the set of all generators of one dimensional cones in Σ . Recall that the toric manifold is obtained as $U(\Sigma)/D(\Sigma)$ where $U(\Sigma) = \mathbb{C}^N \setminus Z(\Sigma)$, where $Z(\Sigma)$ is a closed set related to the primitive collection. Let \mathbb{K} be the subgroup in \mathbb{Z}^N consisting of all lattice vectors $\lambda = (\lambda_1, \dots, \lambda_N)$ such that

$$\begin{aligned} \lambda_1 v_1 + \dots + \lambda_N v_N &= 0. \\ 0 \rightarrow \mathbb{K} \rightarrow \mathbb{Z}^N \xrightarrow{\pi} \mathbb{Z}^n \rightarrow 0, \end{aligned} \quad (2.6)$$

where the map π sends the basis vectors e_i to v_i for $i = 1, \dots, N$. Define $D(\Sigma)$ to be the connected commutative subgroup in $(\mathbb{C}^*)^N$ generated by all one-parameter subgroups

$$\begin{aligned} a_\lambda : \mathbb{C}^* &\rightarrow (\mathbb{C}^*)^N, \\ t &\mapsto (t^{\lambda_1}, \dots, t^{\lambda_N}) \end{aligned}$$

where $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{K}$.

Now, in [8] Yong-Geun Oh and the author, classified all holomorphic discs with boundary on Lagrangian torus fibers. The classification theorem says that such a map can be lifted to a map $D^2 \rightarrow \mathbb{C}^N \setminus Z(\Sigma)$ such that each factor is given by Blaschke products. More precisely, say j -th coordinate is given by

$$c_j \cdot \prod_{k=1}^{\mu_j} \frac{z - \alpha_{j,k}}{1 - \bar{\alpha}_{j,k} z}, \quad \text{for } \alpha_{j,k} \in \text{int}(D^2), \quad c_j \in \mathbb{C}^*, \quad (2.7)$$

where μ_j are non-negative integers for $j = 1, \dots, N$.

Hence, the homotopy classes of holomorphic discs is generated by β_j for $j = 1, \dots, N$ where β_j the homotopy class of the map $(1, \dots, z, \dots, 1)$ with z in j -th entry. Note that its Maslov index is two. Now, we fix a homotopy class β with $\mu(\beta) = 2n$ to consider three point open Gromov-Witten number. We write $\beta = \sum_{k=1}^n \beta_{j_k}$.

We first assume the following condition on the indices $\{j_1, \dots, j_n\}$. Consider $\mathbb{Z}^l \subset \mathbb{Z}^N$, where \mathbb{Z}^l is a subgroup generated by standard vectors e_{j_k} for $k = 1, \dots, n$. Here $l \leq n$ since some j_k can be repeated. Now, we require that the image of \mathbb{Z}^l under the map π in (2.6) is equal to \mathbb{Z}^n . Otherwise, it is easy to show that the boundary images of the holomorphic discs which pass through a point on L lies on a set of codimension one or higher due to the action of $D(\Sigma)$ or the repetition of indices j_k . Hence without the assumption, the three point invariant would be always zero.

The assumption also implies the following. We pick a generic point p_0 , and any lift $\tilde{p}_0 \in \mathbb{C}^N \setminus Z(\Sigma)$. We write $\tilde{p}_0 = (c_1, c_2, \dots, c_N)$. Then, there is a unique lift of L to $\tilde{L} \subset \mathbb{C}^N \setminus Z(\Sigma)$ such that the i -th coordinate of any point in \tilde{L} equals c_i if $i \neq j_k$ for all k . This follows from the assumption that π is an isomorphism when restricted to $\mathbb{Z}^l = \mathbb{Z}^n$.

Hence, we can lift the entire intersection problem to $\mathbb{C}^n \subset \mathbb{C}^N$, and obtain the similar results as the Proposition 2.2.

Note that we carried out the computation with the standard complex structure J_0 , but in [8] it was also shown that J_0 is Fredholm regular. This may be considered as a generic phenomenon. Hence, the “3 -point open Gromov-Witten invariant with a fixed homotopy class” is not well-defined for toric Fano manifolds, namely, it depends on the choice of three points.

Example 2.1. We explain the above computation for the case of $L \subset \mathbb{C}P^2$, where $L = \{[z_0, z_1, z_2] | z_i \in S^1\}$. In this case, there are three homotopy classes of Maslov index 4 discs, $\beta_1 + \beta_2, \beta_0 + \beta_2, \beta_0 + \beta_1$. which contributes to the three point intersection.

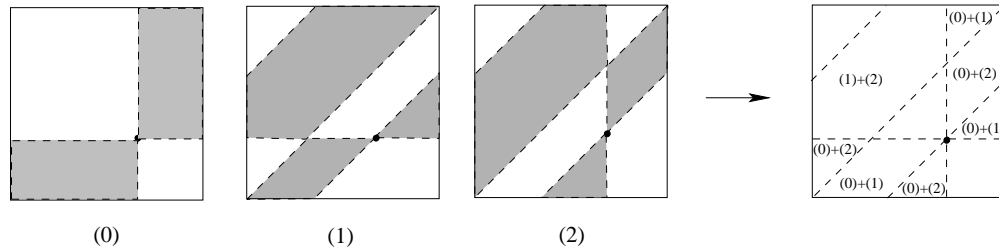


FIGURE 2. The case of Clifford torus in $\mathbb{C}P^2$

We identify L with quotient of the rectangle as in the Figure 2, and we may assume that p_0 corresponds to the vertex. We choose p_1 in the lower right corner of L for simplicity. Then, from the above Proposition, the shaded area in each figure (i) corresponds to the possible position of p_2 so that p_0, p_1, p_2 intersects the Maslov index 4 disc of homotopy class $\beta_{i+1} + \beta_{i+2}$ with indices considered modulo 3.

Surprisingly in this case, the total sum of such holomorphic discs with sign actually vanishes. In the rightmost figure, in each region, the homotopy classes of holomorphic discs which passes through are described and one can notice that either there exists no holomorphic discs of Maslov index 4 at all or they occur in pairs. Unfortunately, this kind of phenomenon does not seem to occur for intersection problems with more than 3 points. Although this section (except this example) is from the preprint arXiv:math/0604502, we learned the vanishing of $\mathbb{Z}/2\mathbb{Z}$ counting from the discussion with Biran and Cornea who considered somewhat different enumerative invariants for T^2 with a different methods in [1].

3. AN EXAMPLE OF OPEN-CLOSED CASE

In this section, we show an example of open-closed Gromov-Witten type invariant considering two different intersection problems together

Theorem 3.1. *Let n_1 be the signed number of J -holomorphic discs of Maslov index 2 in $\mathbb{C}P^n$ with boundary on T^n which intersect a fixed generic hyperplane H at the interior of the disc and a point $p \in T^n$ at the boundary. Let n_2 be the signed number of intersections of the hyperplane H with 2 dimensional chain $Q_p^J \subset L$. Then $n_1 + n_2 = 1$ for any generic choice of hyperplane and for a generic point $p \in T^n$ and for any generic complex structure $J \in \mathcal{J}^{reg}$.*

Remark 3.1. First of all, Q_p^J depends on the choice of p and J , hence this is not a usual enumerative problem. If we do not add the intersection number of the hyperplane with a certain 2-chain Q_p^J , the number of holomorphic disc with the given intersection condition actually changes. See Figure 3: When the hyperplane intersects the interior of the triangle Q_p^J , it does not intersect any holomorphic disc.

We first explain the two dimensional chain Q_p^J which depends on p and J . In [3], we explained that although Bott-Morse Floer cohomology of the Clifford torus is isomorphic to the singular cohomology of the torus, the actual cycles of each homology are different. For example $\partial p = 0$ in the singular cohomology for a point $p \in T^n$, but

$$\delta_{HF}(p) = l_0 + \cdots + l_n,$$

where l_i is the boundary image of the holomorphic disc $[1; \cdots; z; \cdots; 1]$ with $z \in D^2$ in the $(i+1)$ -th entry. Note that $l_0 + \cdots + l_n$ is not zero on the chain level, but is homologous to zero. Hence we can choose a 2-chain $Q_p^{J_0} \subset L$ with $\partial Q_p^{J_0} = -(l_0 + \cdots + l_n)$, then the sum $p + Q_p^{J_0} \otimes T^{\omega(D)}q$ + higher order terms, turns out to be the correct cycle in Floer homology (See [3] for details). In the theorem, the higher order terms do not appear due to a dimensional reason. We also remind the reader that from the Proposition 4.1 of [3], the cycle $p' + Q_{p'} \otimes T^{\omega(D)}q + \cdots$ for another point $p' \in T^2$ is in the same Floer cohomology class as that of p .

In the case of a generic compatible almost complex structure J , we may assume that all the simple J -holomorphic discs are Fredholm regular. In fact, in our case, the Maslov index 2 discs are minimal (cannot bubble off a disc), hence is automatically simple. Hence, we can define the chain Q_p^J for the generic J as above. Also, for a generic family of J_t , $\cup_{t \in [0,1]} Q_p^{J_t}$ can be chosen so that it defines a geometric chain in L whose boundary is $Q_p^{J_1} - Q_p^{J_0}$.

Proof. The idea of the proof is that these holomorphic discs of Maslov index two (up to $PSL(2; \mathbb{R})$ action) can be put together, but they form a chain with a

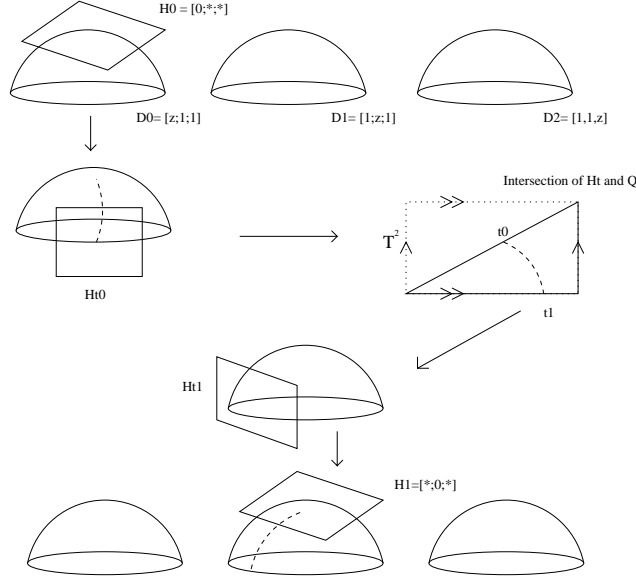


FIGURE 3. A simple example of the cobordism when moving a hyperplane

boundary, which is homologous to zero. And Q is the two dimensional chain whose boundary is exactly the same as that of discs but with an opposite sign. Hence, when they are considered together, they behave like a cycle.

The independence over the choice of the hyperplane H or the point p can be proved similarly and we only show that it is independent over the choice of generic almost complex structure J . Here, we choose J_1, J_2 generic and choose a generic path J_t connecting J_0 and J_1 .

We consider the moduli space $\mathcal{M}_{1,1}(\beta, J_t)$ for some t with $\mu(\beta) = 2$. This is the moduli space of J_t -holomorphic discs of homotopy class β with one interior and one boundary marked points. And we consider the moduli space $\mathcal{M}(\beta, para) = \sqcup_{0 \leq t \leq 1} \{t\} \times \mathcal{M}_{1,1}(\beta, J_t)$. There is an obvious evaluation map $ev = (ev_0^+, ev_0) : \mathcal{M}(\beta, para) \rightarrow (M \times L)$. The preimage $ev^{-1}(H, p)$ provide a cobordism of a counting between the case of J_0 and J_1 . But, there is additional codimension one boundary stratum, obtained as the interior point approaches to ∂D^2 . This may be considered as $H \cap m_1^{J_t} p$ where $m_1^{J_t}$ is the Bott-Morse Floer boundary operator.

Now, we consider the intersection of H with $\cup_{t \in [0,1]} Q_p^{J_t}$. Note that the latter is a 3-dimensional chain. By genericity, the intersections are one dimensional manifolds with boundaries. The possible boundary terms are $H \cap Q_p^{J_0}$, $H \cap Q_p^{J_1}$ or $H \cap \partial Q_p^{J_t}$. The first two terms provide the cobordism between the counting for J_0 and J_1 , and the last term $H \cap Q_p^{J_1}$ will cancel out $H \cap d_1^{J_1} p$ which appeared previously. Here,

$$m_1^{J_t}(p) + m_{1,0}(Q_p^{J_t}) = 0,$$

where $m_{1,0}$ is a part of the Bott-Morse Floer boundary operator which takes a boundary of a chain. In fact the above statement can be verified that it holds with \mathbb{Z} -coefficients. Or one observes that if one proceeds with a given orientation conventions, then one can always adjust the orientation of $Q_p^{J_t}$ so that the above statement holds with sign. \square

The Figure 3 shows an example of a cobordism between two different choices of hyperplanes in $\mathbb{C}P^n$. Note that the middle part of the cobordism is given by the intersection $H_t \cap Q_p^{J_0}$. An easy count is given when a hyperplane is given by $\{[u_0, \dots, u_n] | u_0 = 0\}$ and a disc given by $[z, 1, 1]$ with $J = J_0$.

We remark that chains such as Q_p^J are quite natural as they appear as a part of m_1 -homology cycle of the Floer homology of torus fibers in toric manifolds. Namely, recall from Proposition 4.1 [3] that, if the Floer cohomology is non-vanishing, for any chain P with $\partial P = 0$, there is a sum of chains $\Psi(P)$, which defines a cycle in the Bott-Morse Floer cohomology. In the case of the Clifford torus $T^n \subset \mathbb{C}P^n$, this formula is written as

$$\Psi(P) = P + Q \times P + \frac{1}{2}Q \times Q \times P + \dots + \frac{1}{k}Q \times \dots \times Q \times P, \quad (3.1)$$

where $Q = Q_{p_0}^{J_0}$ is the 2 dimensional chain defined in section 3, with $p_0 = [1, \dots, 1] \in \mathbb{C}P^n$, and k is the largest integer smaller than $(n - \dim(P))/2$.

There is a slight error in the definition 4.4 of [3] where we need to replace $\sum_{i_1 < \dots < i_k}$ by $1/k \sum_{i_1, \dots, i_k}$. And the chains such as $Q \times P$ are defined as geometric chains using by multiplication in $T^n \cong (S^1)^n \subset \mathbb{C}P^n$.

Hence, for example, one may consider product of point homology cycles and its m_2 -product is well-defined homologically. This would in turn imply the invariance property of the sum of several intersection problems involving higher order terms of (3.1). Here invariance is with a fixed J_0 as for $J \neq J_0$, we do not know a precise form of higher order terms of (3.1).

4. CYCLIC A_∞ -ALGEBRAS

In this section we recall the definition of cyclic A_∞ -algebras and also describe another formulation of such structure. For the definition of A_∞ -algebra and its homomorphisms, we refer readers to [12]. We remark that in [10], Fukaya has proved that the A_∞ -algebra of Lagrangian submanifold can be constructed as cyclic filtered A_∞ -algebra.

4.1. Cyclic A_∞ -algebras. The cyclic structure on an A_∞ -algebra was first considered by Kontsevich [16] as an invariant symplectic form on the non-commutative formal manifolds. There are somewhat different sign conventions (whether one is working with degree shifting or not) and we refer readers to [4] or [7] for more details. For simplicity, we consider unfiltered A_∞ -algebras and the filtered case can be dealt in the same way.

Definition 4.1. An A_∞ -algebra $(C, \{m_*\})$ is said to have a *cyclic* inner product if there exists a skew-symmetric non-degenerate, bilinear map

$$\langle, \rangle: C[1] \otimes C[1] \rightarrow R,$$

such that for all integer $k \geq 1$,

$$\langle m_{k,\beta}(x_1, \dots, x_k), x_{k+1} \rangle = (-1)^K \langle m_{k,\beta}(x_2, \dots, x_{k+1}), x_1 \rangle. \quad (4.1)$$

where $K = |x_1|'(|x_2|' + \dots + |x_{k+1}|')$. Here $|x|' = |x| - 1$ is the shifted degree of x . For short, we will call such an algebra, cyclic A_∞ -algebra.

There is a notion of cyclic A_∞ -homomorphism due to Kajiiura [14]

Definition 4.2. An A_∞ -homomorphism $\{h_k\}_{k \geq 1}$ between two cyclic A_∞ -algebras is called a cyclic A_∞ -homomorphism if

- (1) h_1 preserves inner product $\langle a, b \rangle = \langle h_1(a), h_1(b) \rangle$.
 (2)

$$\sum_{i+j=k} \langle h_i(x_1, \dots, x_i), h_j(x_{i+1}, \dots, x_k) \rangle = 0. \quad (4.2)$$

4.2. Other forms of cyclic symmetry. Now, we prove certain equations induced from A_∞ -equations and cyclicity. We first mention a version of Stoke's theorem, which follows immediately from the cyclic symmetry.

Lemma 4.1.

$$\langle m_1(x), y \rangle = (-1)^{|x|} \langle x, m_1(y) \rangle$$

Proof.

$$\langle m_1(x), y \rangle = (-1)^{|x|'|y|'} \langle m_1(y), x \rangle = (-1)^{|x|'|y|' + |x|'(|y|'+1)+1} \langle x, m_1(y) \rangle$$

where the last equality is from the graded skew symmetry of \langle, \rangle . This proves the lemma. \square

Also A_∞ -equation give rise to the following equality.

Proposition 4.2. *Let (A, m_*) be a cyclic A_∞ -algebra. We have the following identity.*

$$0 = \sum_{\sigma, k_1, k_2} (-1)^{Kos} \langle m_{k_1}(x_{\sigma(1)}, \dots, x_{\sigma(k_1)}), m_{k_2}(x_{\sigma(k_1+1)}, \dots, x_{\sigma(k_1+k_2)}) \rangle \quad (4.3)$$

where the summation is over all cyclic permutations $\sigma \in \mathbb{Z}/(k+1)\mathbb{Z}$ and for $k_1+k_2 = k+1$ with additional condition that $1 \in \{\sigma(1), \dots, \sigma(k_1)\}$. Here $(-1)^{Kos}$ is the Koszul sign occuring to rearrange

$$m_{k_1} m_{k_2} x_1 x_2 \dots x_{k+1} \Rightarrow m_{k_1} x_{\sigma(1)} \dots x_{\sigma(k_1)} m_{k_2} x_{\sigma(k_1+1)} \dots x_{\sigma(k_1+k_2)},$$

where we regard m_* as a letter with degree 1 and x_i with degree $|x_i|'$.

Remark 4.3. The additional condition, $1 \in \{\sigma(1), \dots, \sigma(k_1)\}$, make sure that x_1 always appear on the first factor. We may remove this condition and then the resulting equation is just the twice of the above equation.

Proof. The sign $(-1)^{Kos}$ will be used as defined in the above statement.

First, note that in the expression (4.3), the cases when either k_1 or k_2 equals 1 is

$$(-1)^{|x_1|'} \langle m_1(x_1), m_k(x_2, \dots, x_{k+1}) \rangle \quad (4.4)$$

$$(-1)^{Kos} \langle m_k(x_{i+1}, \dots, x_{i-1}), m_1(x_i) \rangle \quad (4.5)$$

for $i = 2, \dots, k+1$.

We start with the Stoke's theorem applied to $m_k(x_1, \dots, x_k)$:

$$\begin{aligned} & - \langle m_1(m_k(x_1, \dots, x_k)), x_{k+1} \rangle \\ & + (-1)^{|m_k(x_1, \dots, x_k)|} \langle m_k(x_1, \dots, x_k), m_1(x_{k+1}) \rangle = 0 \end{aligned} \quad (4.6)$$

Note that $|m_k(x_1, \dots, x_k)| = |x_1|' + \dots + |x_k|'$. Hence, the second term appears in the expression (4.3) with the correct sign, which is in fact the term (4.5) when $i = k+1$.

Now, we apply A_∞ -equation to the first term of (4.2):

$$- \langle m_1(m_k(x_1, \dots, x_k)), x_{k+1} \rangle =$$

$$< m_k(m_1(x_1), \dots, x_k), x_{k+1} > + \dots + (-1)^{Kos} < m_k(x_1, \dots, m_1(x_k)), x_{k+1} > \quad (4.7)$$

$$+ \sum_{k_1+k_2=k+1} (-1)^{Kos} < m_{k_1}(x_1, \dots, x_i, m_{k_2}(x_{i+1}, \dots, x_{i+k_2}), \dots, x_k), x_{k+1} > \quad (4.8)$$

One may check that (4.7) gives rise to the terms in (4.4) and (4.5) with the correct sign. Hence we focus on the expression (4.8). Note that they can be divided into two cases, $i = 0$ or $i \neq 0$.

In the case that $i \neq 0$, by applying the cyclic symmetry, we move the expression $m_{k_2}(x_{i+1}, \dots, x_{i+k_2})$ to the right hand side of the $<, >$:

$$\sum_{\substack{k_1+k_2=k+1 \\ \beta=\beta_1+\beta_2, \beta_i \neq 0}} (-1)^{Kos} < m_{k_1}(x_{i+k_2+1}, \dots, x_{k+1}, x_1, \dots, x_i), m_{k_2}(x_{i+1}, \dots, x_{i+k_2}) > .$$

For the case that $i = 0$, we have

$$\begin{aligned} & \sum (-1)^{Kos} < m_{k_1}(m_{k_2}(x_1, \dots, x_{k_2}), x_{k_2+1}, \dots, x_k), x_{k+1} > \\ &= \sum (-1)^{\epsilon_1} < m_{k_1}(x_{k_2+1}, \dots, x_k, x_{k+1}), m_{k_2}(x_1, \dots, x_{k_2}) > \\ &= \sum (-1)^{\epsilon_2} < m_{k_2}(x_1, \dots, x_{k_2}), m_{k_1}(x_{k_2+1}, \dots, x_k, x_{k+1}) > \\ &= \sum (-1)^{\epsilon_3} < m_{k_1}(x_1, \dots, x_{k_1}), m_{k_2}(x_{k_1+1}, \dots, x_k, x_{k+1}) > . \end{aligned}$$

Here, we have

$$\begin{aligned} \epsilon_1 &= 1 \cdot (|x_{k_2+1}|' + \dots + |x_{k+1}|') + (|x_1|' + \dots + |x_{k_2}|')(|x_{k_2+1}|' + \dots + |x_{k+1}|') \\ \epsilon_2 &= \epsilon_1 + 1 + (|x_1|' + \dots + |x_{k_2}|' + 1)(|x_{k_2+1}|' + \dots + |x_{k+1}|' + 1) \\ &= |x_1|' + \dots + |x_{k_2}|' \\ \epsilon_3 &= Kos \end{aligned}$$

Here, the first equality is from cyclic symmetry and the second equality is from the graded skew symmetry of $<, >$ and the third equality is just relabelling of indices and the resulting sign exactly corresponds to the required the Koszul sign of the formula (4.3). One may check easily that we have produced all the terms in (4.3) with the correct sign.

To prove the statement in the remark that removing the condition on 1 gives rise to the twice of (4.3), note that the case with x_1 on the right hand side of $<, >$ equals the similar term with x_1 on the left hand side of $<, >$ by applying the skew symmetry of $<, >$. In this case these two expressions do not cancel out as the negative sign of skew symmetry of $<, >$ is cancelled out by the negative sign of exchange of two m 's. \square

Conversely, given the maps $\{m_k\}$ with cyclic symmetry (4.1)(which is non-degenerate), the equation (4.3) is equivalent to the A_∞ -equation.

4.3. Hochschild homology of A_∞ -algebra. We recall the definition of Hochschild and cyclic homology of an A_∞ -algebra for reader's convenience. We assume that the base ring of A_∞ -algebra contains \mathbb{Q} . Let $(A, \{m_k\})$ be a filtered A_∞ -algebra. We denote

$$C^k(A, A) = A[1] \otimes C[1]^{\otimes k}.$$

We will denote its degree \bullet part as $C_\bullet^k(A, A)$. We define the Hochschild chain complex

$$C_\bullet(A, A) = \widehat{\bigoplus}_{k \geq 0} C_\bullet^k(A, A), \quad (4.9)$$

after completion with respect to energy filtration and with the boundary operation

$$d^{Hoch} : C_\bullet(A, A) \rightarrow C_{\bullet+1}(A, A)$$

defined as follows: For $v \in A$ and $x_i \in A$,

$$\begin{aligned} d^{Hoch}(\underline{v} \otimes x_1 \otimes \cdots \otimes x_k) = & \sum_{\substack{0 \leq j \leq k+1-i \\ 1 \leq i}} (-1)^{\epsilon_1} \underline{v} \otimes \cdots \otimes x_{i-1} \otimes m_j(x_i, \cdots, x_{i+j-1}) \otimes \cdots \otimes x_k \\ & + \sum_{\substack{0 \leq i, j \leq k \\ i+j \leq k}} (-1)^{\epsilon_2} \underline{n_{i,j}(x_{k-i+1}, \cdots, x_k, v, x_1, \cdots, x_j)} \otimes x_{j+1} \otimes \cdots \otimes x_{k-i} \end{aligned} \quad (4.10)$$

We underline the module elements and the signs follow Koszul convention:

$$\begin{aligned} \epsilon_1 &= |v|' + |x_1|' + \cdots + |x_{i-1}|', \\ \epsilon_2 &= \left(\sum_{s=1}^i |x_{k-i+s}|' \right) (|v|' + \sum_{t=1}^j |x_t|'). \end{aligned}$$

Let us recall cyclic homology of A . For the cyclic generator $t_{n+1} \in \mathbb{Z}/(n+1)\mathbb{Z}$, we define its action on $A^{\otimes(n+1)}$:

$$t_{n+1} \cdot (x_0, x_1, \cdots, x_n) = (-1)^{|x_n|'(|x_0|' + \cdots + |x_{n-1}|')} (x_n, x_0, \cdots, x_{n-1}).$$

Here, we set t_1 to be identity on A and write the identity map as 1. Consider $N_{n+1} := 1 + t_{n+1} + t_{n+1}^2 + \cdots + t_{n+1}^n$.

As in the classical case, we have the natural augmented exact sequence:

$$A^{\otimes(n+1)} \xleftarrow{1-t_{n+1}} A^{\otimes(n+1)} \xleftarrow{N_{n+1}} A^{\otimes(n+1)} \xleftarrow{1-t_{n+1}} A^{\otimes(n+1)} \xleftarrow{N_{n+1}} \cdots$$

We consider $\bigoplus_{n=1}^\infty N_n$ action on $\bigoplus_{n=1}^\infty A^{\otimes n}$ and denote it as

$$N : C_\bullet(A, A) \mapsto C_\bullet(A, A). \quad (4.11)$$

We can also similarly define $(1-t) : C_\bullet(A, A) \mapsto C_\bullet(A, A)$.

The Connes' complex for cyclic homology is defined as

$$(C^\lambda(A, A), d^{Hoch}) = (C_\bullet(A, A)/1-t, d^{Hoch}).$$

From the invariant-coinvariant relation, one can also define cyclic homology by $(C_\bullet(A, A))^{cyc}$ considering invariant elements of cyclic action on $C_\bullet(A, A)$ with bar differential \widehat{d}_{bar} . (see [6] for more details).

Given an element $(\sum_{\sigma \in \mathbb{Z}/k\mathbb{Z}} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)})$ of $(C_\bullet(A, A))^{cyc}$, we associate an element $a_1 \otimes \cdots \otimes a_k$ or $\frac{1}{k}(\sum_{\sigma \in \mathbb{Z}/k\mathbb{Z}} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)})$ of $C^\lambda(A, A)$. The map may be considered as N^{-1} . Here N^{-1} is not well-defined as a map to $C_\bullet(A, A)$ but is well-defined as a map to $C^\lambda(A, A)$.

5. GENERALIZED COUNTING

In this section, we show that cyclic symmetry provides a function on homology theories of A-infinity algebras, namely, function on Hochschild, cyclic homology of A_∞ -algebra or on (cyclic) Chevalley-Eilenberg homology of the induced L_∞ -algebra. (In the preprint of 2006, we called it big cohomology cycles, but they were in fact well-known homology cycles. See [6] for more details).

5.1. Heuristic idea. We first define m^+ operation as in [9].

Definition 5.1. For $k \geq 1$, we define

$$m_k^+(x_1, \dots, x_k) = \langle m_{k-1}(x_1, \dots, x_{k-1}), x_k \rangle \in \Lambda_{nov}.$$

We may sometimes write m_k^+ as m^+ since k may be determined by the inputting element.

Now, we consider the filtered A_∞ -algebra of Lagrangian submanifold defined in [12]. Consider an intersection problem of a holomorphic disc with $k + 1$ marked points intersecting chains x_1, \dots, x_{k+1} at the corresponding marked points. One observes that with the cyclic inner product \langle, \rangle defined from the usual intersection pairing, the number of such holomorphic discs with prescribed intersection may be given by $m^+(x_1, \dots, x_{k+1})$.

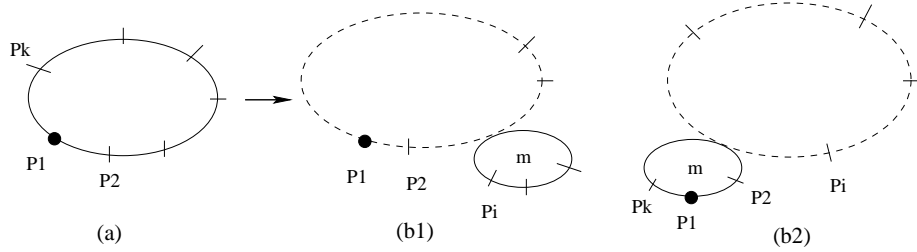


FIGURE 4. Bubbling off and Hochschild boundary operation

Now, consider a moduli space of holomorphic discs with $k + 1$ marked points intersecting chains x_1, \dots, x_{k+1} and its boundary strata from the disc bubblings. In fact, when there exist a disc bubbling, we cannot tell which component is the original or the bubble (say (b1) or (b2)). In fact each diagram can be interpreted in both ways for cyclic A_∞ -algebras. Then twice of such configuration may be interpreted as having an problem with boundary intersection on $d^{Hoch}(x_1, \dots, x_{k+1})$. Here (b1) and (b2) corresponds to the first and third term of (4.10).

Hence the heuristic idea suggests that with cyclicity, intersection with Hochschild cycles do not have codimension one strata from disc bubbling (as they cancel out), and should have an invariant intersection number.

But in fact, the argument above is not entirely correct as it is not yet possible to construct cyclic A_∞ -algebra of singular chains on L directly due to technical issues of transversality, but Fukaya has constructed a cyclic A_∞ -algebra for the differential forms of L . From now on, we consider general cyclic A_∞ -algebras.

5.2. Hochschild homology and cyclic symmetry. We show that m^+ is well-defined on $C_\bullet(A, A)/Im(d^{Hoch})$. When restricted to d^{Hoch} -cycles, we obtain a well-defined function on Hochschild homology of A .

Theorem 5.1. *For a cyclic A_∞ -algebra A , m^+ gives rise to a well-defined function on $C_\bullet(A, A)/Im(d^{Hoch})$. In particular m^+ defines a function on the Hochschild homology of A and on the Chevalley-Eilenberg homology of the underlying L_∞ -algebra \tilde{A}*

$$m^+ : H_\bullet(A, A) \rightarrow \Lambda_{0, nov}, \quad m^+ : H_\bullet^{CE}(\tilde{A}, \tilde{A}) \rightarrow \Lambda_{0, nov}.$$

Proof. For this, we only need to prove that

$$m^+(d^{Hoch}(\underline{x}_0 \otimes \cdots \otimes x_k)) = 0$$

From the definition of Hochschild differential (we omit $(-1)^{Kos}$), LHS equals

$$m^+(\sum_{\substack{0 \leq j \leq k+1-i \\ 1 \leq i}} \underline{x}_0 \otimes \cdots \otimes x_{i-1} \otimes m_j(x_i, \cdots, x_{i+j-1}) \otimes \cdots \otimes x_k) \quad (5.1)$$

$$+ m^+(\sum_{\substack{0 \leq i, j \leq k \\ i+j \leq k}} \underline{m_{i+j+1}(x_{k-i+1}, \cdots, x_k, x_0, x_1, \cdots, x_j)} \otimes x_{j+1} \otimes \cdots \otimes x_{k-i}) \quad (5.2)$$

The expression (5.1) equals (by applying cyclic rotation if necessary)

$$\sum_{\substack{0 \leq j \leq k+1-i \\ 1 \leq i}} \langle m_{k+1-j}(\cdots, x_0, \cdots, x_{i-1}), m_j(x_i, \cdots, x_{i+j-1}) \rangle \quad (5.3)$$

This equals the expression in the Proposition 4.2, hence vanishes.

The expression (5.2) equals

$$\sum_{\substack{0 \leq i, j \leq k \\ i+j \leq k}} \langle m_{k-i-j}(m_{i+j+1}(x_{k-i+1}, \cdots, x_0, \cdots, x_j), x_{j+1}, \cdots, x_{k-i-1}), x_{k-i} \rangle \quad (5.4)$$

$$= \sum_{\substack{0 \leq i, j \leq k \\ i+j \leq k}} \langle m_{k-i-j}(x_{j+1}, \cdots, x_{k-i-1}, x_{k-i}), m_{i+j+1}(x_{k-i+1}, \cdots, x_0, \cdots, x_j) \rangle \quad (5.5)$$

This also vanishes from the Proposition 4.2.

The statement on Chevalley-Eilenberg homology follows from the Hochschild case: if we use the notation

$$[x_1, \cdots, x_k] = \sum_{\tau \in S_k} (-1)^{\epsilon(\tau, \vec{x})} x_{\tau(1)} \otimes \cdots \otimes x_{\tau(k)},$$

an element of Chevalley-Eilenberg chain may be written as $\underline{x}_0 \otimes [x_1, \cdots, x_k]$ and its Chevalley-Eilenberg differential also can be understood as

$$d^{CE}(\underline{x}_0 \otimes [x_1, \cdots, x_k]) = d^{Hoch}(\underline{x}_0 \otimes [x_1, \cdots, x_k]).$$

We refer readers to section 4.1 of [6] for more details. \square

5.3. The case of cyclic homology. Note that for cyclic A_∞ -algebra, we have for any $\sigma \in \mathbb{Z}/(k+1)\mathbb{Z}$,

$$m^+(x_0, \cdots, x_k) = (-1)^{K(\vec{x})} m^+(x_{\sigma(1)}, \cdots, x_{\sigma(k)}). \quad (5.6)$$

From this, we note that m^+ vanishes on $Im(1-t)$. And as cyclic homology is defined by the Connes' complex $(C_\bullet(A, A)/1-t, d_{Hoch})$, the discussion for Hochschild homology implies also that m^+ gives a well-defined function on cyclic homology also. By symmetrization, one obtains similar statement for cyclic Chevalley-Eilenberg homology:

Theorem 5.2. *For an A_∞ -algebra A , m^+ gives rise to well-defined functions on the cyclic homology of A and the cyclic Chevalley-Eilenberg homology of the underlying L_∞ -algebra \tilde{A}*

$$m^+ : HC_\bullet(A) \rightarrow \Lambda_{0, nov}, \quad m^+ : HC_\bullet^{CE}(\tilde{A}) \rightarrow \Lambda_{0, nov}.$$

We remark that the A_∞ -equation can be translated into the following equation

Lemma 5.3. *We have*

$$m^+(\widehat{d}(x_1 \otimes \cdots \otimes x_k) \otimes x_{k+1}) = 0.$$

Proof. Recall that A_∞ -equation may be written as $m(\widehat{d}(x_1 \otimes \cdots \otimes x_k)) = 0$, and the lemma follows from the definition of m^+ . \square

Remark 5.2. One might hope to have

$$m^+(\widehat{d}(x_1 \otimes \cdots \otimes x_{k+1})) = 0.$$

but this does not hold in general. For example, one can easily make an example that $m^+(\widehat{d}(x_1 \otimes \cdots \otimes x_4)) \neq 0$ with the assumption $m_0 = m_1 = 0$.

5.4. Invariance. We explain the invariance of the function m^+ with respect to the cyclic A_∞ -homomorphism. Let $(A, m_A, \langle, \rangle_A)$, $(B, m_B, \langle, \rangle_B)$ be two cyclic A_∞ -algebras and let $h : A \rightarrow B$ be an A_∞ -homomorphism. Then, from h one can define the cohomomorphism \widehat{h} , and an induced map $\widehat{h} : C_\bullet(A, A) \rightarrow C_\bullet(B, B)$ which is a chain map between two Hochschild chain complexes.

Proposition 5.4. *If $h : A \rightarrow B$ is a cyclic A_∞ -homomorphism, it preserves the value of m^+ . Namely, for a Hochschild cycle (or chain) α , we have*

$$m_A^+(\alpha) = m_B^+(\widehat{h}(\alpha))$$

Proof. The proof follows from the definition of a cyclic A_∞ -homomorphism (Definition 4.2). Namely

$$\begin{aligned} m_B^+(\widehat{h}(\cdots)) &= \langle m_B(h(\cdots), h(\cdots), \cdots, h(\cdots)), h(\cdots) \rangle_B \\ &= \langle h(\cdots, m_A(\cdots), \cdots), h(\cdots) \rangle_B = \langle h_1(m_A(\cdots)), h_1(\cdot) \rangle_B \\ &= \langle m_A(\cdots), \cdot \rangle_A \end{aligned}$$

Here, first equality follows from the definition of \widehat{h} , and the second equality is A_∞ -equation on the LHS, and the third and the last equalities are from the definition of cyclic A_∞ -homomorphism. \square

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